Approximate Analytical Solution of a Nonlinear Boundary Value Problem and its Application in Fluid Mechanics

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Although the decomposition method and its modified form were used during the last two decades by many authors to investigate various scientific models, a little attention was devoted for their applications in the field of fluid mechanics. In this paper, the Adomian decomposition method (ADM) is implemented for solving the nonlinear partial differential equation (PDE) describing the peristaltic flow of a power-law fluid in a circular cylindrical tube under the effect of a magnetic field. The numerical solutions obtained in this paper show the effectiveness of Adomian's method over the perturbation technique.

Key words: Adomian Decomposition Method; Power-Law Fluid.

1. Introduction

The Adomian decomposition method (ADM) in applied mathematics is an effective procedure to obtain analytic and approximate solutions for different types of operator equations [1 – 17]. It is based on the search for a solution in the form of a series. In this paper, we consider the following non-dimensional boundary value problem (BVP):

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\partial w}{\partial r}\right)^n\right] = -\frac{\mathrm{d}p}{\mathrm{d}z} - M^2 w \tag{1}$$

with the boundary conditions

$$\begin{aligned} \frac{\partial w}{\partial r} &= 0 \qquad \text{at } r = 0, \\ w &= -1 \qquad \text{at } r = h(z). \end{aligned} \tag{2}$$

The nonlinear partial differential equation (1) with the boundary conditions (2) often occurs in the fluid flow problems of a power law fluid in a circular cylindrical tube when a travelling wave is imposed to the boundary under the assumptions of long wave length and low Reynolds number with an external force (magnetic field) [18]. The unknown function $w(r,z)$ represents the axial velocity component of the fluid particles and n is the index of the power law fluid. $\frac{\mathrm{d}p}{\mathrm{d}z}$ is the pressure gradient term where $p = p(z)$. The second term in the right hand side represents the external force (magnetic field) acting on the fluid. M is the non-dimensional parameter of the magnetic field (Hartmann number). In [18] a regular perturbation series in terms of the dimensionless Hartmann number M has been used to obtain an analytic solution assuming that M is a small parameter. In fact, we can use ADM to obtain the analytic solution without any such restrictions on the Hartmann number M. So it is our objective in this paper to show how to apply Adomian's method to obtain the analytic solution for the nonlinear PDE (1) with the boundary conditions (2) without any restrictions on M.

2. Direct Approach

In this section, we give a direct approach to solve (1) with the boundary conditions (2). Firstly, we rewrite (1) in the operator form

$$L_\mathrm{r} w = -\frac{\mathrm{d}p}{\mathrm{d}z} - M^2 w, \tag{3}$$

where the differential operator L_r (nonlinear operator) is defined in the form

$$L_\mathrm{r}[.] = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\partial}{\partial r}[.]\right)^n\right]. \tag{4}$$

The proposed approach depends mainly on the Adomian decomposition method but with a new definition for the inverse operator L_r^{-1}:

$$L_r^{-1}[.] = -\int_h^r \sqrt[n]{r^{-1}\int_0^r r[.]\,\mathrm{d}r}\ \mathrm{d}r. \tag{5}$$

Applying this inverse operator to the left-hand side of (1), we obtain

$$\begin{aligned} &L_r^{-1}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\partial w}{\partial r}\right)^n\right]\right) \\ &= -\int_h^r \sqrt[n]{r^{-1}\int_0^r r\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\partial w}{\partial r}\right)^n\right]\right)\mathrm{d}r}\ \mathrm{d}r \\ &= -\int_h^r \sqrt[n]{r^{-1}\left[r\left(-\frac{\partial w}{\partial r}\right)^n\right]}\ \mathrm{d}r \\ &= \int_h^r \frac{\partial w}{\partial r}\,\mathrm{d}r \\ &= w(r,z) - w(h,z) \\ &= w(r,z) + 1, \end{aligned} \tag{6}$$

where the two boundary conditions (2) are used directly. Now, operating with L_r^{-1} on (1), it then follows:

$$w(r,z) = -1 + L_r^{-1}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} - M^2 w\right). \tag{7}$$

Notice that L_r^{-1} is a nonlinear operator. Therefore

$$\begin{aligned} w(r,z) &= -1 - \int_h^r \sqrt[n]{-\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}r - M^2 r^{-1}\int_0^r rw\ \mathrm{d}r}\ \mathrm{d}r \\ &= -1 - \int_h^r \sum_{m=0}^{\infty} A_m. \end{aligned} \tag{8}$$

Where A_m are Adomian polynomials for the nonlinear term,

$$f(w) = \sqrt[n]{-\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}r - M^2 r^{-1}\int_0^r rw\ \mathrm{d}r},$$

and can be found from the formula [1]:

$$A_m = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left[f\left(\sum_{i=0}^{\infty}\lambda^i w_i\right)\right]_{\lambda=0},\quad m \ge 0. \tag{9}$$

The standard Adomian's method defines the solution $w(r,z)$ by the series $w = \sum_{m=0}^{\infty} w_m$, consequently (8) can be written as

$$\sum_{m=0}^{\infty} w_m = -1 - \int_h^r \sum_{m=0}^{\infty} A_m\ \mathrm{d}r, \tag{10}$$

hence, the solution $w(r,z)$ can be computed by using the recurrence relation

$$\begin{aligned} w_0 &= -1, \\ w_m &= -\int_h^r A_m\ \mathrm{d}r,\ \ m \ge 1. \end{aligned} \tag{11}$$

To find w_1, we use formula (9) to form A_0 as

$$\begin{aligned} A_0 &= \sqrt[n]{-\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}r - M^2 r^{-1}\int_0^r rw_0\ \mathrm{d}r} \\ &= \left(\frac{1}{2}\right)^{\frac{1}{n}}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)^{\frac{1}{n}} r^{\frac{1}{n}}. \end{aligned} \tag{12}$$

Therefore,

$$w_1 = \frac{n}{n+1}\left(\frac{1}{2}\right)^{\frac{1}{n}}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)^{\frac{1}{n}}\left(h^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}\right). \tag{13}$$

Using formula (9) again to generate A_1, we get

$$\begin{aligned} A_1 &= -\frac{M^2}{n}\left(-\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}r - M^2 r^{-1}\int_0^r rw_0\ \mathrm{d}r\right)^{\frac{1}{n}-1} \\ &\quad\cdot\left(r^{-1}\int_0^r rw_1\ \mathrm{d}r\right) \\ &= -\left(\frac{1}{2}\right)^{\frac{2}{n}}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)^{\frac{2}{n}-1}\frac{M^2}{(n+1)(3n+1)} \\ &\quad\cdot\left[(3n+1)h^{\frac{1}{n}+1}r^{\frac{1}{n}} - 2nr^{\frac{2}{n}+1}\right]. \end{aligned} \tag{14}$$

Consequently,

$$\begin{aligned} w_2 &= \left(\frac{1}{2}\right)^{\frac{2}{n}}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)^{\frac{2}{n}-1}\frac{nM^2}{(n+1)^2(3n+1)} \\ &\cdot\left[(3n+1)h^{\frac{1}{n}+1}r^{\frac{1}{n}+1} - (2n+1)h^{\frac{2}{n}+2} - nr^{\frac{2}{n}+2}\right]. \end{aligned} \tag{15}$$

Using the first three components w_0, w_1, and w_2, then the series solution is given by

$$\begin{aligned} w(r,z) &= -1 + \frac{n}{n+1}\left(\frac{1}{2}\right)^{\frac{1}{n}}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)^{\frac{1}{n}} \\ &\quad\cdot\left(h^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}\right) + \left(\frac{1}{2}\right)^{\frac{2}{n}}\left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)^{\frac{2}{n}-1} \\ &\quad\cdot\frac{nM^2}{(n+1)^2(3n+1)}\left[(3n+1)h^{\frac{1}{n}+1}r^{\frac{1}{n}+1}\right. \\ &\quad\left. - (2n+1)h^{\frac{2}{n}+2} - nr^{\frac{2}{n}+2}\right] + \ldots. \end{aligned} \tag{16}$$

Of course, it is possible to calculate more components in the decomposition series (16) to enhance the approximation. Also, it is important to note that the series solution given by (16) is obtained by ADM without any restrictions on the parameter M, consequently the range of applicability of M becomes more wider than in perturbation in which $0 < M < 1$. To make this point as clear as possible, we discuss in Section 5 the effectiveness of ADM in finding numerical solutions with good accuracy for (1) when $n = 1$, in which the exact solution is known and $M \geq 1$.

3. Exact Solution at $M = 0$

In this case, the exact solution can be derived from (16) as

$$w = -1 - \frac{n}{n+1}\left(-\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}\right)^{\frac{1}{n}}\left(h^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}\right), \tag{17}$$

which is the exact solution of (1) with the boundary conditions (2) in the absence of external force, i.e., $M = 0$.

4. Exact Solution at $n = 1$ (Newtonian Fluid), $M \neq 0$

For $n = 1$, (1) with the boundary conditions (2) has the exact solution

$$w(r,z) = -1 + \left(1 - \frac{1}{M^2}\frac{\mathrm{d}p}{\mathrm{d}z}\right)\left[1 - \frac{I_0(Mr)}{I_0(Mh)}\right]. \tag{18}$$

Where $I_0(Mr)$ is the modified Bessel function of first kind. Setting $n = 1$ in (16), we obtain the series solution as

$$\begin{aligned} w = &-1 + \left(-\frac{\mathrm{d}p}{\mathrm{d}z} + M^2\right)\left[\frac{h^2}{4} - \frac{3M^2h^4}{64}\right. \\ &\left. + \left(-\frac{1}{4} + \frac{M^2h^2}{16}\right)r^2 - \frac{M^2}{64}r^4\right] + \dots. \end{aligned} \tag{19}$$

In fact, this series solution represents the first few terms of the Taylor expansion for the exact solution given by (18). Furthermore, it is obtained without any restrictions on M, so the range of applicability for M is more wider than in the perturbation solution, this point is indicated numerically in the next section.

5. Numerical Results and Discussion

In order to verify numerically whether the Adomian's methodology leads to accurate solutions, nu-

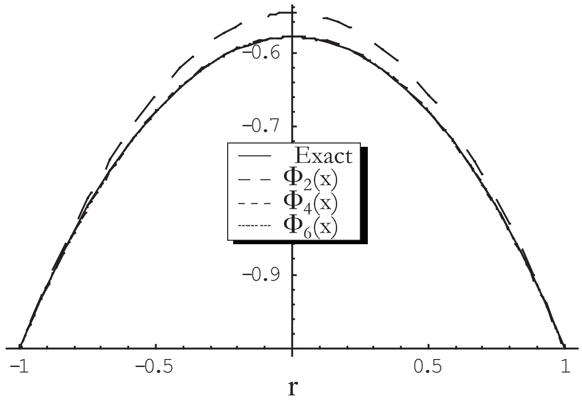

Fig. 1. Comparison between Adomian's approximate solutions and the exact one at $n = 1$, $h = 1$, $\mathrm{d}p/\mathrm{d}z = -1$, $M = 1$.

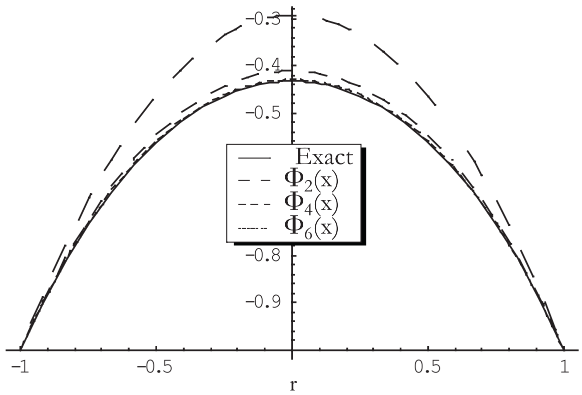

Fig. 2. Comparison between Adomian's approximate solutions and the exact one at $n = 1$, $h = 1$, $\mathrm{d}p/\mathrm{d}z = -1$, $M = 1.5$.

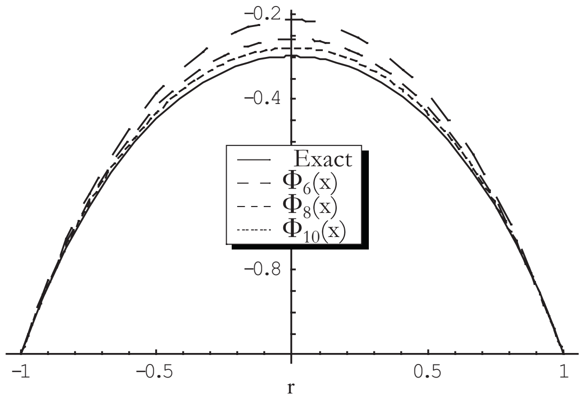

Fig. 3. Comparison between Adomian's approximate solutions and the exact one at $n = 1$, $h = 1$, $\mathrm{d}p/\mathrm{d}z = -1$, $M = 2$.

merical calculations have been carried out using the software package Mathematica5. Using the series solution given by (19), we plot the approximate solutions obtained by Adomian's method with the exact solution given by (18) at $h = 1$, $\frac{dp}{dz} = -1$ and for $M = 1, 1.5, 2$ in Figures 1–3, respectively. The numerical results in all the figures show that a good approximation is achieved using small values of m-terms of the decomposition series solution, $\Phi_m = \sum_{i=0}^{m-1} w_i$. It is also seen from these figures that as M increases, more terms of the decomposition series are needed to achieve a good approximation. Finally, we observe that the solution obtained by ADM is already valid for any M, while in perturbation it is valid only for $0 < M < 1$, of course it is one of the main advantages of Adomian's method.

6. Conclusion

In this paper, a relatively new analytical technique, the Adomian decomposition method, is implemented for solving a nonlinear PDE of special interest in fluid mechanics. The solution obtained in this paper is found to be valid for any Hartmann number M. Of course it is one of the main advantages of the decomposition method over the other techniques.

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